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1987 J. Phys. A: Math. Gen. 20 4321

(http://iopscience.iop.org/0305-4470/20/13/033)

J. Phys. A: Math. Gen. 20 (1987) 4321-4330. Printed in the UK

Entropy and the central limit theorem in quantum mechanics

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Received 14 July 1986

Abstract. We give a short proof that any infinitely divisible even representation of a Clifford algebra or a CAR algebra is quasi-free, and of Hudson's central limit theorem for fermions. We show that the entropy of any even state of the CAR, Clifford or CCR algebra is less than or equal to the entropy of the quasi-free state with the same two-point function.

1. Introduction

The concepts of infinitely divisible cyclic representations of a group [1], Lie algebra [2] and Clifford algebra [3] were introduced in the references given and the related concept of factorisable representations for groups was independently formulated in [4]. When the group or Lie algebra is Abelian, we get infinitely divisible random variables [5] as a special case. We can then construct a stochastic process with independent increments or a generalised random field with independent 'values' at every point [6]. Another special case is the class of 'ultralocal' representations of the canonical commutation relations [7] first constructed in a similar way by Araki [8]. For a general Lie algebra, we get a representation of current algebra [1, 2, 4, 7]. Here, the infinite divisibility is equivalent to conditional positivity analogous to that known in probability theory [5], theorem A 1.1. This is better expressed in terms of group cocycles [4] or Lie algebra cocycles [2]. A similar construction can be given for associative algebras [9].

Non-trivial cocycles exist for various groups [4, 10] and we get examples of Gaussian and some non-Gaussian boson quantum fields. For a nice survey of the mathematical questions, see [11].

Infinitely divisible representations of a Clifford algebra or a CAR algebra lead to the construction of ultralocal fermion fields [3], but we showed that only quasi-free fields can be obtained in this way, because all infinitely divisible even representations of the CAR algebra are quasi-free [3], theorem 2.7. Some people hope that this rather limiting result would not apply to the more general fermionic structures of [9, 12, 13].

In this paper we give a straightforward proof of this result (also for the real case, which was omitted in [3]). It is a consequence of the boundedness of the fermion field, and the additivity of the cumulants for independent fields [3]. The latter follows easily from the properties of the 'generating element', i.e. generating function with values in a Grassmann algebra [14-16]. This allows a generalisation of the concept of ∞-divisibility to superalgebras. A nice unified treatment of bosons and fermions can be given in terms of Hopf products [17, 18].

Hudson's central limit theorem [19] also follows easily. Using this, and also the boson version [20], we show that the entropy of the quasi-free state ω_Q, with the same two-point functions as a state ω, is greater than or equal to the entropy of ω.

2. Preliminaries

Let $\mathcal{H}$ be a real Hilbert space, and $\mathcal{P}(\mathcal{H})$ the associative non-commutative polynomial algebra over $\mathcal{H}$. We impose the relations

$$ab+ba-\langle a, b\rangle_{\mathcal{H}}1=0 \qquad a, b\in\mathcal{H} \tag{2.1}$$

on $\mathcal{P}(\mathcal{H})$, i.e. we form the quotient of $\mathcal{P}$ by the two-sided ideal J generated by the LHS of (2.1). Note that addition and real scalar multiplication in $\mathcal{H}$ agree with formal addition and real scalar multiplication in $\mathcal{P}(\mathcal{H})$ for elements of $\mathcal{H}$, i.e. $\mathcal{H}\to\mathcal{P}(\mathcal{H})$ is an $\mathbb{R}$-linear injection. We take complex scalars for $\mathcal{P}$. Finally we make $\mathcal{P}/J=Cl(\mathcal{H})$ into a *-algebra by defining $\lambda=\bar\lambda$ if $\lambda\in\mathbb{C}$, $a^*=a$ for $a\in\mathcal{H}$, and extend $*$ by induction to $Cl(\mathcal{H})$ from $(\lambda A)^*=\bar\lambda A^*$, $(AB)^*=B^*A^*$ when $\lambda\in\mathbb{C}$, $A, B\in Cl(\mathcal{H})$. This $*$-algebra is the Clifford algebra over $\mathcal{H}$.

There is a parallel definition of the algebra of the canonical anticommutation relations over a complex Hilbert space $\mathcal{H}$, denoted CAR($\mathcal{H}$). For this, let $a\to\bar a$ be the (antilinear) identification of $\mathcal{H}$ with its dual, $\mathcal{H}^*$. In $\mathcal{P}(\mathcal{H}\oplus\mathcal{H}^*)$, for which the injection $\mathcal{H}\to\mathcal{P}(\mathcal{H}\oplus\mathcal{H}^*)$ is $\mathbb{C}$-linear and $\mathcal{H}^*\to\mathcal{P}(\mathcal{H}\oplus\mathcal{H}^*)$ is antilinear, we impose the relations

$$ab+ba=0 \qquad a\bar b+\bar b a=\langle b, a\rangle_{\mathcal{H}} \qquad a, b\in\mathcal{H}$$

(and here we use the physicists' convention for scalar product-linear in the second variable). Finally we introduce a $*$-operation, by extending $(a)^*=\bar a$, etc.

If $\mathcal{H}$ is a real or complex Hilbert space, $\mathfrak{A}(\mathcal{H})$ will denote the Clifford or CAR algebra over $\mathcal{H}$. The subspace $\mathcal{H}\subseteq$ CAR($\mathcal{H}$) is the space of creation operators and the subspace $\mathcal{H}^*\subseteq$ CAR($\mathcal{H}$) is the space of annihilation operators.

A *cyclic representation* of $\mathfrak{A}$ is a triple (π, Ω, K) where K is a Hilbert space, $\Omega\in K$ is cyclic (i.e. $\pi(\mathfrak{A})\Omega$ is dense in K) and π is a homomorphism from $\mathfrak{A}$ to bounded operators on K. Two cyclic representations $(\omega_1, \Omega_1, K_1)$ and $(\omega_2, \Omega_2, K_2)$ are said to be cyclic equivalent if there exists a unitary map $W: K_1$ onto K_2 such that $W\Omega_1=\Omega_2$ and $W\pi_1=\pi_2 W$.

$\mathfrak{A}(\mathcal{H})$ is Z_2-graded, i.e. is a direct sum of even and odd parts, with obvious rules for products. A cyclic representation (π, Ω, K) is said to be even if $\langle\Omega, \pi(A)\Omega\rangle=0$ for all odd A. If (π, Ω, K) is even, then $K=K_o\oplus K_e$, where K_o contains the odd vectors and K_e the even. On such a space the parity operator

$$\beta=-1_{K_o}\oplus 1_{K_e}$$

is well defined, and $\beta\pi(a)=-\pi(a)\beta$, $a\in\mathcal{H}$.

Suppose (π_1, Ω_1, K_1) and (π_2, Ω_2, K_2) are two even cyclic representations of $\mathfrak{A}(\mathcal{H}_1)$ and $\mathfrak{A}(\mathcal{H}_2)$. Then we can construct a cyclic representation π, of $\mathfrak{A}(\mathcal{H}_1\oplus\mathcal{H}_2)$, acting on $K_1\otimes K_2$, with cyclic vector $\Omega_1\otimes\Omega_2$, by

$$\pi(a_1\oplus a_2)=\pi_1(a_1)\otimes 1_{K_2}+\beta_{K_1}\otimes\pi_2(a_2) \qquad a_1, a_2\in\mathcal{H}_1, \mathcal{H}_2.$$

The action of π on a general polynomial is then determined by the requirement that π is a $*$-homomorphism.

This construction [3], written $\pi = \pi_1 \wedge \pi_2$, sometimes called the Chevalley product [17], is the natural analogue of the tensor product for cyclic group representations, which in turn is the generalisation of adding independent random variables. The Chevalley product has some natural associativity, e.g. $\pi_1 \wedge (\pi_2 \wedge \pi_3)$ and $(\pi_1 \wedge \pi_2) \wedge \pi_3$ are cyclic equivalent.

If we put $\mathscr{H}_1 = \ldots = \mathscr{H}_k = \mathscr{H}$ we can restrict a product representation $\pi = \pi_1 \wedge \ldots \wedge \pi_k$ to the diagonal $\{a \oplus a \oplus \ldots \oplus a \colon a \in \mathscr{H}\}$. We do not quite get a representation of $\mathfrak{A}(\mathscr{H})$, but what we called a k-representation [3]; namely $a \to k^{-1/2}\pi(a) \wedge \ldots \wedge \pi(a)$ (k-factors) restricted to the cyclic space generated from the product of the cyclic vector, is a cyclic representation, called π^k.

Definition [3]. We say a representation (π, Ω, K) of $\mathfrak{A}(\mathscr{H})$ is ∞-divisible if for every integer $k > 0$, there exists a cyclic representation (ϕ, Y, L) such that π is cyclic equivalent to ϕ^k.

We now define the generating functional [13, 14]. For $Cl(\mathscr{H})$ we furnish $\mathscr{H} = \mathscr{H}^*$ with a Grassmann multiplication. We will use the symbol $\eta \in \mathscr{H}$ when its Grassmann nature is meant. We assume η anticommutes with $a \in \mathscr{H}$, $\mathscr{H}$ regarded as a subset of $Cl(\mathscr{H})$.

Let (a_j) be an orthonormal basis in $\mathscr{H}$ and let (η_j) be the components of η in this basis. Then define the Grassmann element associated to (π, Ω, K) by:

$$G_\pi(\eta) = \langle \exp \mathrm{i}\pi(a_k)\eta_k \rangle \in \Lambda(\mathscr{H}).$$

Here, we sum over repeated indices and $\langle\cdot\rangle$ denotes $\langle \Omega, \cdot\, \Omega\rangle$; $\Lambda(\mathscr{H})$ is the fermion Fock space over $\mathscr{H}$, i.e. the Hilbert space of antisymmetric tensors.

We see that the nth moment $\langle \pi(a_1) \ldots \pi(a_n)\rangle$ is the coefficient of $(1/n!)\eta_1\eta_2 \ldots \eta_n$ in the formal power-series expansion of G. Convergence is no worry, since any given nth moment can be evaluated using only the relevant n-dimensional subspace of $\mathscr{H}$.

For CAR$(\mathscr{H})$ we introduce a Grassmann multiplication in $\mathscr{H}$ and in $\mathscr{H}^*$, with elements denoted η or $\bar{\xi}$. These anticommute with each other and with the odd elements of CAR$(\mathscr{H})$. Then define

$$G(\eta, \bar{\xi}) = \langle \exp \mathrm{i}(\pi(a_j)\bar{\xi}_j + \pi(\bar{a}_j)\eta_j)\rangle \in \Lambda(\mathscr{H} \oplus \mathscr{H}^*)$$

where (a_j) is an orthonormal basis in $\mathscr{H}$ and $(\bar{a}_j)$ is the corresponding basis in $\mathscr{H}^*$.

Then if $G_1, \ldots, G_k$ are the characteristic elements of representations $\pi_1, \ldots, \pi_k$ of $a(\mathscr{H}_1), \ldots, a(\mathscr{H}_k)$, then the Grassmann product $G_1 G_2 \ldots G_k$ is the characteristic element of $\pi_1 \wedge \ldots \wedge \pi_k$ [15].

The Grassmann element $\log G$ generates the truncated functions, which are the analogues of the cumulants. Clearly, the cumulants add under Chevalley product. (This result was proved in [3].) Specialising to the diagonal we see that the cumulants of the k-representation $\pi \wedge \pi \wedge \ldots \wedge \pi$ (k-factors) are k times the cumulants of π. This is the result we need.

3. The quasi-free nature of infinitely divisible representations

Theorem [3]. Let (π, Ω, K) be an ∞-divisible even representation of $\mathfrak{A} = Cl(\mathscr{H})$ or CAR$(\mathscr{H})$. Then π is quasi-free in that $\log G(\eta)$ is a (positive-definite) quadratic form in η (for $Cl(\mathscr{H})$) or $\log G(\eta, \bar{\xi})$ is a (positive-definite) sequilinear form. Conversely, every quasi-free representation has this form and is ∞-divisible.

Proof. We show all cumulants of order >2 vanish. The positive-definiteness of the second cumulant follows from that of the second moment, to which it is equal if the representation is even.

For any representation π, the operators $\pi(a)$, (or $\pi(a)$, $\pi(\bar{a})$ for $\mathrm{CAR}(\mathscr{H})$) are bounded by $\|a\|$. Hence if $\|a_j\| = 1$, $j = 1, \ldots, m$, the mth moment obeys

$$|\langle \pi(a_1) \ldots \pi(a_m)\rangle| \leqslant 1. \tag{3.1}$$

Denote the moments by $\langle a_1 \ldots a_m\rangle$ and the cumulants by $\langle a_1 \ldots a_m\rangle_{\mathrm{T}}$; both these are homogeneous of degree m. Now, the moments are written as a sum (over partitions) of products of cumulants

$$\langle a_1 \ldots a_m\rangle = \{a_1 \ldots a_m\rangle_{\mathrm{T}} + \sum_{\text{partitions}} \pm\langle a_1 \ldots\rangle_{\mathrm{T}} \ldots \langle \ldots a_m\rangle_{\mathrm{T}}.$$

The partitions of $(1 \ldots m)$ form a lattice, and we can form the (Möbius) inversion and write $\langle a_1 \ldots a_m\rangle_{\mathrm{T}}$ as a sum of products of moments of order $\leqslant m$. Let $\# m$ denote the sum of the absolute values of all the (integer) coefficients in this formula. It follows from (3.1) that for any cyclic representation π of $\mathfrak{A}$

$$|\langle \pi(a_1) \ldots \pi(a_m)\rangle_{\mathrm{T}}| \leqslant \# m.$$

Now suppose π is ∞-divisible. Then for any n there is ϕ such that

$$\pi \simeq n^{-1/2}(\phi \wedge \phi \wedge \ldots \wedge \phi) = \phi^n.$$

The additivity of cumulants then says that the cumulants of $\phi \wedge \ldots \wedge \phi$ are n times those of ϕ, and the homogeneity says that the mth cumulant of ϕ^n is $(1/\sqrt{n})^m$ times that of $\phi \wedge \ldots \wedge \phi$, and is thus $n(1/\sqrt{n})^m$ times that of ϕ:

$$\begin{aligned}\langle \pi(a_1) \ldots \pi(a_m)\rangle_{\mathrm{T}} &= n(1/\sqrt{n})^m \langle \phi(a_1) \ldots \phi(a_m)\rangle_{\mathrm{T}} \\ &\leqslant (\# m) n^{(2-m)/2} \qquad \text{for any } n.\end{aligned}$$

Letting $n \to \infty$ gives

$$\langle \pi(a_1) \ldots \pi(a_m)\rangle_{\mathrm{T}} = 0 \qquad m > 2.$$

The converse is obvious.

Hudson's central limit theorem also follows at once from the cumulant addition theorem: replace π by π^n; this does not change the second moments. The mth cumulant of π^n is bounded by $n(1/\sqrt{n})^m \times m$th cumulant of ρ. This goes to zero as $n \to \infty$. Thus π^n converges to the even quasi-free representation with the same two-point functions. This convergence is strong over finite-dimensional subsets of $\mathscr{H}$.

4. The quasi-free reduction for fermions

Given any even state ω on $\mathfrak{A}(\mathscr{H})$, we can define ω_Q to be that quasi-free state with the same second moments. We show that $\omega \to \omega_Q$ is the best approximation to ω, in the sense that ω_Q is the state of greatest entropy among all states with the same second moments as ω.

Consider, then, $\mathfrak{A}(\mathscr{H})$, first where $\dim \mathscr{H} < \infty$, and let (π, Ω, K) be a cyclic representation. The expectation $\langle \Omega, \pi(A)\Omega\rangle$ of any $A \in \mathfrak{A}(\mathscr{H})$ can be written in terms of a density matrix ϕ_Ω on $\Lambda\mathscr{H}$ as

$$\langle \Omega, \pi(A)\Omega\rangle = \mathrm{Tr}(\phi_\Omega A)$$

where Tr is the trace over $\Lambda\mathscr{H}$. ϕ_Ω is uniquely determined by (π, Ω) since in the Fock representation the $\mathfrak{A}(\mathscr{H})$ generate all bounded operators. We then define the entropy S_π of the cyclic representation (π, Ω, K) to be

$$S_\pi = -\mathrm{Tr}(\phi_\Omega \log \phi_\Omega).$$

Clearly, this is the same for any (cyclically) equivalent representation. Since the entropy of a tensor product of density matrices is the sum of the entropies of each factor, we have $S_{\pi_1 \wedge \pi_2} = S_{\pi_1} + S_{\pi_2}$. Given a density matrix ϕ on $K_1 \otimes K_2$, we obtain the marginal states by partial traces $\mathrm{Tr}_2\, \phi$ and $\mathrm{Tr}_1\, \phi$. Here, Tr_j means taking the trace over the factor K_j. Then [22] the entropies of ϕ are less than or equal to the sum of the entropies of the two marginal states.

Theorem 1. Let (π, Ω, K) be an even cyclic representation of $\mathfrak{A}(\mathscr{H})$, where possibly $\dim \mathscr{H} = \infty$. Then $\pi^2 = 2^{-1/2}\pi \wedge \pi$ does not have less entropy than π, when both are restricted to any subalgebra $\mathfrak{A}(\mathscr{H}_0)$ with $\mathscr{H}_0 \subseteq \mathscr{H}$, and $\dim \mathscr{H}_0 < \infty$.

Proof. Restrict to $\mathscr{H}_0$. Let π_1, π_2 be independent copies of π. Then $K \otimes K$ carries a representation of $\mathfrak{A}(\mathscr{H}_0 \oplus \mathscr{H}_0)$ based on the cyclic vector $\Omega \otimes \Omega$. Moreover, the term $\Lambda(\mathscr{H}_0 \oplus \mathscr{H}_0) = \Lambda(\mathscr{H}_0) \otimes \Lambda(\mathscr{H}_0)$ carries the Fock representation π_{F} of $\mathfrak{A}(\mathscr{H}_0 \oplus \mathscr{H}_0)$ and $\Omega \otimes \Omega$ is represented by the density matrix $\phi_\Omega \otimes \phi_\Omega$.

Now perform the rotation in $\mathscr{H}_0 \oplus \mathscr{H}_0$ given by

$$\left.\begin{aligned} a_j' &= (a_j + 1 + 1 + a_j)/\sqrt{2} \\ a_j'' &= (a_j' - 1 - 1 - a_j)/\sqrt{2} \end{aligned}\right\} \qquad j = 1, \ldots, \dim \mathscr{H}_0$$

where (a_j), $j = 1, \ldots, \dim \mathscr{H}_0$ is a basis in $\mathscr{H}_0$, and (a_j', a_j'') is the new basis in $\mathscr{H}_0 \oplus \mathscr{H}_0$. Let (a_j') span $\mathscr{H}'$ and (a_j'') span $\mathscr{H}''$. Then $\Lambda(\mathscr{H}_0 \oplus \mathscr{H}_0) = \Lambda(\mathscr{H}') \otimes \Lambda(\mathscr{H}'')$, though $\phi_\Omega \otimes \phi_\Omega$ does not factor when we write $\Lambda(\mathscr{H}_0) \otimes \Lambda(\mathscr{H}_0)$ in this way. Now restrict the cyclic representation $(\pi \wedge \pi, \Omega \otimes \Omega, \Lambda(\mathscr{H} \oplus \mathscr{H}))$ to the subalgebras $\mathfrak{A}(\mathscr{H}')$ and $\mathfrak{A}(\mathscr{H}'')$, to get the marginal representations π', π'' with density matrices

$$\phi' = \mathrm{Tr}_{\Lambda(\mathscr{H}'')}(\phi_\Omega \otimes \phi_\Omega)$$

and

$$\phi'' = \mathrm{Tr}_{\Lambda(\mathscr{H}')}(\phi_\Omega \otimes \phi_\Omega).$$

By the evenness of Ω, (π', ϕ') and (π'', ϕ'') are cyclically equivalent, and so have the same entropy. Hence

$$2S(\pi) = S(\pi \wedge \pi) \leqslant S(\pi') + S(\pi'') = 2S(\pi').$$

Thus $S(\pi') \geqslant S(\pi)$.

Corollary. The quasi-free reduction $\omega \to \omega_Q$ is entropy non-decreasing since, by Hudson's theorem, $\pi \to \pi^2 \to \pi^4 \to \ldots$ converges (on any $\mathfrak{A}(\mathscr{H}_0)$, with $\dim \mathscr{H}_0 < \infty$) to a quasi-free state. The dimension of the representation space $\Lambda(\mathscr{H}_0)$ is finite, so the entropy of the limit is the limit of the entropies, which increase. The second moments are conserved at each step, so the limit is π_Q, the GNS representation from ω_Q.

5. The quasi-free reduction for bosons

Let $\mathscr{H}$ be a real Hilbert space of even or infinite dimension, and let σ be a non-degenerate symplectic form on $\mathscr{H}$. The restriction of σ to $\mathscr{H}_0$, an even finite-dimensional subspace of $\mathscr{H}$, might be degenerate; but there always exists an even finite-dimensional subspace $\mathscr{L} \supseteq \mathscr{H}_0$ on which σ is non-degenerate. For each such $\mathscr{L}$, let $a \to W(a)$ be the Schrödinger representation of the Weyl relations over $\mathscr{L}$:

$$W(a)\, W(b) = \mathrm{e}^{\mathrm{i}\sigma(a,b)}\, W(a+b)$$

a, $b \in \mathscr{L}$, and let $\mathfrak{A}(\mathscr{L})$ be the W^*-algebra generated by these $\{W(a)\colon a \in \mathscr{L}\}$. These form an inductive system, and we follow Segal [23] in defining the C^*-algebra of the CCR over $(\mathscr{H}, \sigma)$ to be the inductive limit of $\{\mathfrak{A}(\mathscr{L}), \mathscr{L} \subseteq \mathscr{H}, \dim \mathscr{L} < \infty, \sigma$ non-degenerate on $\mathscr{L}\}$. Call it $\mathfrak{A}(\mathscr{H})$.

Let ω be a regular state on $\mathfrak{A}(\mathscr{H})$, in the sense of [22]; that is, for each $\mathscr{L}$ as above, the GNS representation of ω restricted to $\mathfrak{A}(\mathscr{L})$ gives a Weyl system over $(\mathscr{L}, \sigma)$, i.e. a strongly continuous representation of the Weyl relations. Then by the Stone–von Neumann theorem, $\{a \to W(a), a \in \mathscr{L}\}$ is a direct sum of Schrödinger representations, and ω is given by a density matrix ϕ_ω on the Fock space $\Gamma(\mathbb{C}^l)$ where $l = \dim_{\mathbb{C}} \mathscr{L}_{\mathbb{C}}$. We define the $\mathscr{L}$-entropy of ω to be

$$S_{\mathscr{L}}(\omega) = -\mathrm{Tr}(\phi_\omega \log \phi_\omega)$$

the trace being taken over $\Gamma(\mathbb{C}^l)$. We say that one regular state ω_1 does not have less entropy than another, ω_2, if $S_{\mathscr{L}}(\omega_1) \geqslant S_{\mathscr{L}}(\omega_2)$ for every even finite-dimensional $\mathscr{L} \subseteq \mathscr{H}$ on which σ is non-degenerate.

Let ω be an even regular state, and let ω_Q be the quasi-free state with the same two-point functions as ω, assumed finite. This section is devoted to proving that $S(\omega_Q) \geqslant S(\omega)$. The method of proof is to use the central limit theorem to show that ω_Q is the W^* limit of a sequence $\{\tau^n \omega\}$ of transformations τ^n of ω; we show that τ is entropy non-decreasing, and that entropy is continuous on the set of states including the limit. The result then follows.

Cushen and Hudson [20] prove a quantum mechanical central limit theorem for one degree of freedom. For more than one degree of freedom, Quaegebeur [21] has given a proof, apparently unaware of the earlier work of Cushen and Hudson. We quote without proof the following version.

Theorem 2. Let ω be an even regular state of $\mathfrak{A}(\mathscr{H})$ with $\dim \mathscr{H} < \infty$, and let $a \to W(a)$ be the corresponding Weyl equation. Let $\mathscr{H}_1$, $\mathscr{H}_2$ be the two orthogonal copies of $\mathscr{H}$ and let $W(a \oplus b) = W(a) \otimes W(b)$ with cyclic vector $\omega \otimes \omega$ be the product Weyl system. Let

$$\tilde{W}(a \oplus b) = W\left\{\frac{a+b}{\sqrt{2}} \oplus \frac{a-b}{\sqrt{2}}\right)$$

a, $b \in \mathscr{H}$ and let $\tau\omega$ be the state defined by the Weyl system $(\tilde{W}(a \oplus 0), \omega \otimes \omega)$ restricted to the cyclic subspace generated from $\omega \otimes \omega$ by acting with $\tilde{W}(a \oplus 0)$, $a \in \mathscr{H}$.

Then $\tau^n \omega$ converges W^* to the quasi-free state ω_Q with the same two-point functions as ω.

We shall also use the following theorem, proved in [24, 25].

Theorem 3. Let N be a self-adjoint operator on a Hilbert space K with spectrum

$\{0, 1, 2, \ldots\}$ and multiplicity $m(j)$ for eigenvalue j, bounded by

$$m(j) \leqslant Kj^d \qquad \text{for some } K, d > 0.$$

Let $T(E)$ be the set of positive trace-class operators ϕ with $\operatorname{Tr}(\phi N) \leqslant E$. Then

$$S(\phi) = -\operatorname{Tr}(\phi \log \phi) = \mathrm{O}(\operatorname{Tr} \phi \log \operatorname{Tr} \phi) \tag{5.1}$$

uniformly in $T(E)$ as $\operatorname{Tr} \phi \to 0$.

We note that, writing $W(a) = \mathrm{e}^{\mathrm{i}Ra}$ for a Weyl system W, the transformation τ of theorem 2 can can be written $\tau\omega(W(a)) = (W'(a))$, with $W'(a) = \mathrm{e}^{\mathrm{i}R'a}$, where the transformation $R \to R'$ is given by

$$R'a = (R_1 a + R_2 a)/\sqrt{2}$$

where R_1 and R_2 are two cyclic equivalent and independent copies of R. On can easily check that all the two-point functions of W' are the same as those of W, if these are finite. It follows that all the states $\{\tau^n\omega\}_{n=0,1,\ldots}$ have the same two-point functions, so that if $\omega(N) = E$, then

$$\tau^n\omega\left(\sum_{i=1}^{l} a_i^* a_i\right) = E$$

for all n. Here, $l = \dim \mathscr{H}$, and $a_1, \ldots, a_l$ is an orthonormal basis in $\mathscr{H}$. We note that the particle number $N = \sum_{i=1}^{l} a_i^* a_i$ has spectrum and multiplicity obeying the conditions of theorem 3, and so $\tau^n\omega$ and ω_Q give rise to density matrices ρ obeying (5.1).

Theorem 4. $S(\omega_Q) = \lim_{n\to\infty} S(\tau^n\omega)$.

Proof. Since $\dim \mathscr{H} < \infty$, the dimension of the subspace $\Gamma(n_0)$ of $\Gamma(\mathscr{H})$ with particle number $\leqslant n_0$ is finite. Choose ε; choose n_0 large enough so that $\operatorname{Tr}_{\Gamma(n_0)\perp}(\phi) < \varepsilon$, for all $\phi \in T(E)$. This is possible since, in a basis in which N is diagonal,

$$\sum j \operatorname{Tr}(P_j\phi) = E$$

where P_j projects onto eigenvalue $N = j$, so

$$\sum_{n_0+1}^{\infty} j \operatorname{Tr}(P_j\phi) < E \qquad \text{and} \qquad n_0 \operatorname{Tr}_{\Gamma(n_0)\perp}(\phi) = n_0 \sum_{n_0+1}^{\infty} \operatorname{Tr}(P_j\phi) < \sum_{n_0+1}^{\infty} j \operatorname{Tr}(P_j\phi) < E.$$

Therefore

$$S(\omega_Q) = S\left(\sum^{n_0} P_j\omega_Q\right) + S\left(\sum_{n_0+1}^{\infty} (P_j\omega_Q)\right) = \lim_{n\to\infty} S\left(\sum^{n_0} P_j\tau^n\omega\right) + S\left(\sum_{n_0+1}^{\infty} (P_j\omega_Q)\right)$$

since S is continuous on finite-dimensional spaces

$$= \lim_{n\to\infty} S(\tau^n\omega) + 0(\varepsilon \log \varepsilon) + 0(\varepsilon \log \varepsilon).$$

Since this is true for every $\varepsilon > 0$, we get $S(\omega_Q) = \lim_{n\to\infty} S(\tau^n\omega)$.

Theorem 5. $S(\omega_Q) > S(\omega)$, with equality if and only if $\omega = \omega_Q$. The 'if' part is obvious.

Remark. The first result in this direction is due to Wichmann [26].

Proof. It is enough, by theorem 4, to show that $S(\tau\omega) > S(\omega)$, with equality if and only if $\omega = \omega_Q$. Now the 'if' part is not quite obvious. We prove it because of its independent interest. Thus let ω be a regular even state of the CCR over H, with

$\dim H<\infty$. Let H_1 and H_2 be two copies of H and form the independent systems over H_1 and H_2, i.e. let $W(a\oplus b)=W(a)\otimes W(b)$ with cyclic vector $\omega\otimes\omega$, represented by a product density matrix on $\Gamma(H_1\oplus H_2)$.

Let

$$\tilde{W}(a\oplus b)=W\left(\frac{a+b}{\sqrt{2}}\oplus\frac{a-b}{\sqrt{2}}\right)$$

with cyclic vector $\omega\otimes\omega$, be a new Weyl system over $H_1\oplus H_2$ $(a\in H, b\in H)$, and let $W'(a)=\tilde{W}(a\oplus 0)$, $W''(a)=\tilde{W}(0\oplus a)$, restricted to the cyclic subspace generated from $\omega\otimes\omega$. Note that W', W'' are cyclic equivalent, since ω is even and that $\tau\omega$ is the state giving either of these Weyl systems by the GNS construction.

Suppose first that ω is quasi-free, of mean 0. Then

$$\omega(W(a))=\exp(-\tfrac{1}{2}\langle a, Qa\rangle)$$

where Q is a quadratic form on $\mathscr{H}$. Then

$$\begin{aligned}(\omega\otimes\omega)(W(a\oplus b)) &=\omega(W(a))\omega(W(b))\\ &=\exp(-\tfrac{1}{2}\langle a, Qa\rangle-\tfrac{1}{2}\langle b, Qb\rangle)\\ &=\exp\left(-\frac{1}{2}\left\langle\frac{a+b}{\sqrt{2}}, Q\frac{a+b}{\sqrt{2}}\right\rangle-\frac{1}{2}\left\langle\frac{a-b}{\sqrt{2}}, Q\frac{a-b}{\sqrt{2}}\right\rangle\right)\\ &=(\omega\otimes\omega)(\tilde{W}(a\oplus b)).\end{aligned}$$

Putting $b=0$ gives $\omega(W(a))=\omega\otimes\omega(W'(a))=\tau\omega(W(a))$, so $\tau\omega=\omega$ if ω is quasi-free. Hence, *a fortiori*, $S(\tau\omega)=S(\omega)$.

For the converse, suppose ω is any even regular state, and let $S(\omega)$, $S'(\omega)$, $S''(\omega)$ be the entropies of the states given by the Weyl systems W, W', W'', on the cyclic space generated from $\omega\otimes\omega$. We have

$$\begin{aligned}2S(\omega)=S(\omega\otimes\omega)\leqslant S'(\omega)+S''(\omega)&=2S'(\omega)\\ &=2S(\tau\omega).\end{aligned}$$

Here we use the fact that the entropy of a density matrix on $\Gamma(\mathscr{H}_1\oplus\mathscr{H}_2)=\Gamma(\mathscr{H}_1)\otimes\Gamma(\mathscr{H}_2)=\Gamma(\mathscr{H}')\otimes\Gamma(\mathscr{H}'')$ is less than or equal to the sum of the two marginal entropies. Since $S(\tau^n\omega)\geqslant S(\omega)$ for all n we have $\lim_{n\to\infty}S(\tau^n\omega)\geqslant S(\omega)$; therefore $S(\omega_Q)\geqslant S(\omega)$, by theorem 4.

It remains to show that if $S(\omega_Q)=S(\omega)$, then $\omega=\omega_Q$. It is enough to show that if $S(\tau\omega)=S(\omega)$ then $\omega=\omega_Q$. Indeed, the inequality $2S\leqslant S'+S''$ is an equality only if W' and W'' are independent, i.e. $W(a\oplus b)=W'(a)\otimes W''(b)$ and

$$(\omega\otimes\omega)(\tilde{W}(a\oplus b))=\omega'(W'(a))\omega''(W''(b)).$$

Then let $f(a)=\omega(W(a))$, $g(a)=(\omega\otimes\omega)(W(a\oplus 0))$. Then, since

$$(\omega\otimes\omega)\left[W\left(\frac{a+b}{\sqrt{2}}\oplus\frac{a-b}{\sqrt{2}}\right)\right]=f\left(\frac{a+b}{\sqrt{2}}\right)f\left(\frac{a-b}{\sqrt{2}}\right)$$

we have, if W', W'' are independent:

$$g(a)g(b)=f\left(\frac{a+b}{\sqrt{2}}\right)f\left(\frac{a-b}{\sqrt{2}}\right)$$

for all a, $b \in \mathscr{H}$. In particular $g(a)^2 = f(\sqrt{2}a)$, i.e. $f(a) = g(a/\sqrt{2})^2$. This gives the functional equation for x, $y \in \mathscr{H}$:

$$g(x)g(y) = \left[g\left(\frac{x+y}{2}\right)\right]^2\left[g\left(\frac{x-y}{2}\right)\right]^2 \qquad g(0) = 1.$$

Since ω is regular, g is continuous (on finite-dimensional spaces $\mathscr{H}$) and so near the origin a branch of $G(x) = \log g(x)$ can be defined such that

$$G(x) + G(y) = 2\left[G\left(\frac{x+y}{2}\right) + G\left(\frac{x-y}{2}\right)\right]. \tag{5.2}$$

Now (5.2) is the Appolonius equality and so G is a quadratic form, i.e. ω is Gaussian so $\omega = \omega_Q$. This proves theorem 5. This is a quantum version of Bernstein's theorem [27]. A similar result under stronger assumptions was proved by Lindsay [28].

The proof of Bernstein's theorem given in [27] cannot be taken over immediately since it assumes x, $y \ldots$ are real variables rather than elements of a complex Hilbert space. There is a version of Bernstein's theorem in the even more general setting of states on a Borchers algebra as described in [9]. Namely, if A and B are independent identical quantised fields such that $A + B$ and $A - B$ are independent, then A (and so B) is quasi-free. I thank G C Hegerfeldt for discussions on this result. There is a similar easy result for the CAR and Clifford cases.

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